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Function Spaces and Algebras

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: Hlavním cílem této práce je rozhodnout, kdy je prostor funkcí ekvivalentní algebře, tj. kdy je uzavřený na bodové násobení funkcí. Nejprve je uvedena teorie určitých prostorů funkcí, konkrétně Lebesgueovy L^p prostory, třída Banachových prostorů funkcí, Banachovy prostory funkcí invariantní vůči nerostoucímu přerovnání, Morreyovy prostory, Campanatovy prostory a prostor slabé-L^∞. Poté je dokázána nutná podmínka k tomu, aby byl prostor funkcí ekvivalentní algebře. Dále je dokázána také postačující podmínka. V každé z těchto dvou podmínek hraje klíčovou roli prostor L^∞. Jako důsledek dále získáme charakterizaci, kdy je Banachův prostor funkcí ekvivalentní algebře. Poté je uvedeno několik příkladů, které ilustrují možné využití získaných výsledků. Následně je uvážen speciální případ těch Banachových prostorů funkcí, které jsou invariantní vůči nerostoucímu přerovnání. Nakonec je otázka, kdy je prostor funkcí ekvivalentní algebře, zodpovězena pro prostory uvedené na začátku.

Klíčová slova: algebry, Banachovy prostory funkcí, prostory funkcí, normy invariantní vůči nerostoucímu přerovnání, vnoření

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Abstract: The primary purpose of this thesis is to determine when a function space is equivalent to an algebra, that is, when it is closed with respect to pointwise multiplication. Firstly, the theory of some function spaces, namely Lebesgue L^p spaces, the class of Banach function spaces, rearrangement-invariant Banach function spaces, Morrey spaces, Campanato spaces, and weak$-L^\infty$, is introduced. Secondly, a general necessary condition, as well as a general sufficient condition, for a function space to be equivalent to an algebra is given. In each of these two conditions, a crucial role is played by the space L^∞. Furthermore, as a corollary, a characterisation when a Banach function space is equivalent to an algebra is obtained. Thereafter, a few examples illustrating possible usage of these results are presented. After that, a special case when a Banach function space is rearrangement invariant is dealt with. Lastly, the matter of equivalence to an algebra is addressed for the function spaces introduced before.

Keywords: Algebras, Banach function spaces, Embeddings, Function spaces, Rearrangement-invariant norms

Contents

Chapter 1

Introduction

Given a function space, or, more generally, some class of functions, one cannot in general expect that this structure will be closed with respect to pointwise multiplication (assuming it makes sense). On the other hand, the question when this is true is of interest in mathematical analysis and its applications, and many authors, both classical and recent, were interested in this question (see e.g. [1], [12], [17]). Some authors investigated particular function classes in this connection ([1, Theorem 4.39] for classical Sobolev spaces, [2] and [7] for Sobolev-Orlicz spaces). In connection with a fairly complicated class of Sobolev-type spaces, this question was recently studied in [8].

In this thesis, we shall address this question in the scope of rather general concept of Banach function spaces (see Section 2.2), Morrey and Campanato spaces (see Section 2.5), and the space weak $-$ L^∞ (see Section 2.6). In Chapter 2, the function spaces just mentioned are formally defined and some of their fundamental properties, which will be needed later in Chapter 3, are stated without proofs. Moreover, the key definition of an algebra, which makes the vague statement "to be closed with respect to pointwise multiplication" precise, is provided (see Definition 2.6). This text is not supposed to be a comprehensive treatment of these spaces. Hence only the properties which will be directly needed in Chapter 3 are listed. Other essential attributes of them (for instance completeness) can be found in many classical books covering the matter of functional spaces (e.g. [4] or [14]). We begin Chapter 3 with proving a sufficient condition and a necessary condition for a function space to be equivalent to an algebra (see Theorem 3.1 and Theorem 3.2). These results indicate that particularly the space of essentially bounded functions (the space L^∞) plays a crucial role. Combining these two conditions together makes us able to characterise when a Banach function space is equivalent to an algebra (see Corollary 3.5). This characterisation answers the question which was raised at the very beginning for many function spaces, for a wide range of function classes is covered by the abstract system of Banach function spaces. We give some simple examples that demonstrate how this characterisation can be used. Afterwards, we focus on Morrey and Campanato spaces and also provide a characterisation when these spaces are equivalent to an algebra (see Theorem 3.12 and Theorem 3.13). In the course of the proof, we show that the space of Hölder functions (see Definition 2.2) is equivalent to an algebra. At the end, we prove that the space weak $-$ L^∞ is not closed with respect to pointwise multiplication (see Theorem 3.15).

Chapter 2

Preliminaries

In this chapter, we shall begin with some fundamental definitions and a notation and then we shall introduce some function spaces which will be considered later.

Convention. In this entire thesis, (R, μ) will once and for all denote a $\sigma-$finite measure space.

Convention. We follow the convention, which is usual in related branches of mathematics, that $0 \cdot \infty = \infty \cdot 0 = 0$.

Notation 2.1. *Assume* (R, μ) *is a* $\sigma-$*finite measure space. We denote*

$\mathfrak{M}(R, \mu) = \{f; f$ *is* μ*-measurable function on* R *whose values are in* $[-\infty, \infty]\}$,
$\mathfrak{M}^+(R, \mu) = \{f \in \mathfrak{M}(R, \mu); f \geq 0\}$.

If no confusion is possible, we will simply write $\mathfrak{M}$ *or* $\mathfrak{M}^+$. *We also denote*

$$\mathfrak{M}_0 = \{f \in \mathfrak{M}; f \text{ is finite } \mu\text{-a.e.}\},$$
$$\mathfrak{M}_0^+ = \{f \in \mathfrak{M}_0; f \geq 0\}.$$

Convention. Throughout this text, we shall often work with subsets of $\mathfrak{M}(R, \mu)$ identifying any two functions which coincide $\mu-$a.e., as usual.

We shall define the space of Hölder functions which generalise Lipschitz functions.

Definition 2.2. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$. $C(\Omega)$ denotes the set of all functions defined and continuous on Ω. We also set

$$C\left(\overline{\Omega}\right) = \{f \in C(\Omega); f \text{ is bounded and uniformly continuous on } \Omega\}.$$

We say that a function f defined on Ω satisfies the *Hölder condition* with exponent λ $(\lambda > 0)$ if there exists a nonnegative constant C (which may depend on f) such that the inequality

$$|f(x) - f(y)| \leq C\,|x - y|^\lambda$$

holds for all $x, y \in \Omega$. The set of all functions $f \in C\left(\overline{\Omega}\right)$ which satisfy the Hölder condition with exponent λ is denoted by $C^\lambda\left(\overline{\Omega}\right)$. We equip $C^\lambda\left(\overline{\Omega}\right)$ with a norm

$$\|f\|_\lambda = \|f\|_\infty + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\lambda}, \quad f \in C^\lambda\left(\overline{\Omega}\right),$$

where

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|.$$

Remark 2.3. One can easily observe that a function which satisfies the Hölder condition with exponent $\lambda > 1$ is constant on Ω.

Given a function space $X \subseteq \mathfrak{M}(R,\mu)$, it is natural to expect that if a function f is contained in the space and a function g is "smaller" than f, then g is also an element of X. This property of function spaces is formalised by the following definition.

Definition 2.4. Let $X \subseteq \mathfrak{M}(R,\mu)$ be a (quasi-)normed linear space. We say that X has the *lattice property* if the following implication

$$|g| \le |f| \ \ \mu-\text{a.e.} \Rightarrow g \in X \text{ and } \|g\|_X \le \|f\|_X$$

holds for all $f \in X$ and $g \in \mathfrak{M}$.

The information when a function space is contained in another function space is definitely of interest. Moreover, if it is, we wish the inclusion to behave well with respect to the topology of the larger space so that, e.g., the convergence of sequences is preserved.

Definition 2.5. Let X and Y be (quasi-)normed linear spaces. We say that X *is embedded into* Y (a fact which will be denoted by $X \hookrightarrow Y$) if $X \subseteq Y$ and there exists a positive constant C such that the inequality

$$\|f\|_Y \le C\|f\|_X$$

holds for all $f \in X$.

We say that X and Y *are equivalent* (and write $X \rightleftarrows Y$) if $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

Now, we are about to define the key term of the entire thesis - the concept of algebras. Even though the theory of algebras can be developed in a much more general fashion, we will not do it here. We shall simply use a definition which will cover the scope of this work. A more general approach is adopted, for instance, in [15] or [16, Chapter 18].

Definition 2.6. We say that a (quasi-)normed linear space $X \subseteq \mathfrak{M}(R,\mu)$ *is equivalent to an algebra* if $fg \in X$ for all $f, g \in X$ and there exists a positive constant C such that for all $f, g \in X$ the estimate

$$\|fg\|_X \le C\|f\|_X\|g\|_X \tag{2.1}$$

holds.

We shall now introduce some important function spaces on which we will focus in the subsequent chapter. We also list some fundamental properties of the spaces which will be needed later. Proofs of the results which will be stated and more information on the spaces can be found, for example, in [4] or [14].

2.1 Lebesgue L^p spaces

The well-known Lebesgue L^p spaces are arguably textbook examples of function spaces. Throughout this work, we will extensively use them, especially L^∞, and therefore we shall define them precisely at first.

Definition 2.7. Let $p \in (0, \infty]$. For $f \in \mathfrak{M}_0(R, \mu)$, we define

$$\|f\|_{L^p(R,\mu)} = \begin{cases} \left(\int_R |f|^p \, d\mu\right)^{\frac{1}{p}} & \text{if } p \in (0, \infty) \\ \operatorname*{ess\,sup}_{x \in R} |f(x)| & \text{if } p = \infty, \end{cases}$$

where $\operatorname*{ess\,sup}_{x \in R} |f(x)| = \inf\{C \geq 0; \mu(\{x \in R; |f(x)| > C\}) = 0\}$. Set

$$L^p(R, \mu) = \{f \in \mathfrak{M}_0(R, \mu); \|f\|_{L^p(R,\mu)} < \infty\}.$$

The set $L^p(R, \mu)$ equipped with the (quasi-)norm $\|\cdot\|_{L^p(R,\mu)}$ is called a *Lebesgue $L^p(R, \mu)$ space.*

The functional $\|\cdot\|_{L^p(R,\mu)}$ is, however, not always a norm on $L^p(R, \mu)$ as the remark below clarifies.

Remark 2.8. If $p \in [1, \infty]$, then $\|\cdot\|_{L^p(R,\mu)}$ is indeed a norm on the set $L^p(R, \mu)$. This is, however, not the case when $p \in (0, 1)$. In this case, $\|\cdot\|_{L^p(R,\mu)}$ fails to be a norm but it is a quasinorm on $L^p(R, \mu)$.

Remark 2.9. When (R, μ) is a completely atomic $\sigma-$finite measure space, it is customary to use the notation $\ell^p(R, \mu)$ instead of $L^p(R, \mu)$. In the common case when $R = \mathbb{N}$ and μ is the counting measure on $\mathbb{N}$, we usually briefly write ℓ^p. The ℓ^p spaces are sometimes alternatively called the sequence spaces.

2.2 Banach function spaces

Now, we shall collect some common properties of many function spaces (including the Lebesgue L^p spaces just defined). The spaces having these properties are covered under the general umbrella of the axiomatic system of Banach function spaces.

Definition 2.10. We say that a mapping $\rho : \mathfrak{M}^+(R, \mu) \to [0, \infty]$ is a *Banach function norm* if the following seven statements hold for all $f, g, f_n \in \mathfrak{M}^+$, $n \in \mathbb{N}$, for all $\alpha \in [0, \infty)$ and for all $\mu-$measurable subsets $E \subseteq R$ such that $\mu(E) < \infty$:

1. $\rho(f) = 0 \Leftrightarrow f = 0 \ \mu-$a.e.,
2. $\rho(\alpha f) = \alpha \rho(f)$,
3. $\rho(f + g) \leq \rho(f) + \rho(g)$,
4. $g \leq f \ \mu-$a.e. $\Rightarrow \rho(g) \leq \rho(f)$,
5. $f_n \uparrow f \ \mu-$a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$,

6. $\rho(\chi_E) < \infty$

7. $\int_E f \, d\mu \leq C_E \rho(f)$,

where C_E is a positive constant independent of f.

For all $f \in X$ where $X = \{f \in \mathfrak{M}; \rho(|f|) < \infty\}$, we define

$$\|f\|_X = \rho(|f|).$$

The pair $(X, \|\cdot\|_X)$ (or briefly just X) is called a *Banach function space.*

Example 2.11. The Lebesgue L^p ($p \in [1, \infty]$) spaces or Orlicz spaces (the definition of Orlicz spaces can be found in [14, Chapter 4]) are typical examples of Banach function spaces.

2.3 Rearrangement-invariant spaces

The standard p-norm ($p \in [1, \infty)$) on $\mathbb{R}^n$ ($\|(x_1, \ldots, x_n)\|_p = \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$) depends simply on the magnitudes of the components of a vector but it does not take the arrangement of the components into account. We shall generalise this property to the abstract concept of Banach function spaces.

The distribution of the values of a measurable function on an arbitrary measure space is represented by its distribution function, which is defined right below. It measures "how large" (by means of the underlying measure) are so-called level sets of the function.

Definition 2.12. We define the *distribution function* μ_f of a function $f \in \mathfrak{M}_0(R, \mu)$ as

$$\mu_f(\lambda) = \mu\left(\{x \in R; |f(x)| > \lambda\}\right), \quad \lambda \geq 0.$$

Remark 2.13. Given a function $f \in \mathfrak{M}_0(R, \mu)$, the sets $\{x \in R; |f(x)| > \lambda\}$ are sometimes called level sets (of f). The reason for this terminology is evident.

As stated at the very beginning of this section, we wish to single out those Banach function spaces whose norms remain constant for functions which have the same distribution of their values. Such functions are said to be equimeasurable.

Definition 2.14. Assume (R, μ) and (S, ν) are $\sigma-$finite measure spaces. Let $f \in \mathfrak{M}_0(R, \mu)$ and $g \in \mathfrak{M}_0(S, \nu)$. We say that f and g are *equimeasurable* if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.

Now, we are ready to proceed to the definition of those Banach function spaces whose norms do not differ among equimeasurable functions.

Definition 2.15. We say that a Banach function norm $\rho : \mathfrak{M}^+(R, \mu) \to [0, \infty]$ from Definition 2.10 is *rearrangement invariant* if $\rho(f) = \rho(g)$ for each pair of equimeasurable functions $f, g \in \mathfrak{M}_0^+(R, \mu)$. The corresponding Banach function space is called a *rearrangement-invariant Banach function space.*

After Definition 2.17 and Remark 2.19 (5), it should be obvious why such Banach function spaces are called rearrangement invariant.

Remark 2.16. The Banach function spaces from Example 2.11 are rearrangement invariant. On the other hand, the Morrey space which shall be defined later (see Definition 2.29) is an example of a Banach function space which is not rearrangement invariant.

2.4 The spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$

If we restrict ourselves to so-called resonant measure spaces (see Definition 2.20), there exist the smallest and the largest (in the sense which will be made precise shortly) rearrangement-invariant Banach function spaces.

Definition 2.17. Let $f \in \mathfrak{M}_0(R, \mu)$. The *nonincreasing rearrangement* of the function f is the function f^* defined as

$$f^*(t) = \inf\{\lambda \geq 0; \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$

The operator $f \mapsto f^*$ has, however, some drawbacks. Firstly, a substantial amount of information can be lost when we pass to f^*. Indeed, let $f(x) = 1 - \mathrm{e}^{-x}$ for $x \in (0, \infty)$. One can readily verify that $f^*(t) = 1$ for all $t \geq 0$ (with respect to the Lebesgue measure on $(0, \infty)$). Arguably even more severe difficulty with f^* is that the operator $f \mapsto f^*$ is not subadditive. Consider, for example, $f = \chi_{[0,1)}$ and $g = \chi_{[1,2)}$ on $\mathbb{R}$. On the one hand, we can easily compute that $(f+g)^*(t) = 1$ for $t \in [0, 2)$ and therefore $(f+g)^*(1) = 1$. On the other hand, clearly $f^*(1) = g^*(1) = 0$. Hence the pointwise estimate $(f+g)^* \leq f^* + g^*$ does not hold. Fortunately, a partial remedy for this issue is the following pointwise estimate

$$(f+g)^*(s+t) \leq f^*(s) + g^*(t) \tag{2.2}$$

which holds for each $s \geq 0$ and $t \geq 0$ (for a proof, see [4, Chapter 2, Proposition 1.7]). It immediately follows from the estimate (2.2) that

$$(f+g)^*(t) \leq f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right)$$

holds for every $t \geq 0$.

Another remedy is provided by the concept of maximal functions, which we shall define now.

Definition 2.18. Let $f \in \mathfrak{M}_0(R, \mu)$. We define the *maximal function* f^{**} of f^* as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(y)\, dy, \quad t > 0.$$

Remarks 2.19.

1. We use the convention that $\inf \emptyset = \infty$.

2. μ_f, f^* as well as f^{**} may attain the value ∞.

3. If $\mu(R) < \infty$, then $f^*(t) = 0$ for all $t \geq \mu(R)$. In this case, we may consider f^* to be a function defined only on the interval $[0, \mu(R))$.

4. Unlike the operator $f \mapsto f^*$, the operator $f \mapsto f^{**}$ is subadditive as can be proven.

5. f and f^* are equimeasurable.

An important integral inequality is the following estimate (due to G. H. Hardy and J. E. Littlewood)

$$\int_R |fg| \, d\mu \leq \int_0^\infty f^*(t) \, g^*(t) \, dt, \tag{2.3}$$

which holds for each $f, g \in \mathfrak{M}_0(R, \mu)$ (a proof can be found in [4, Chapter 2 ,Theorem 2.2]). An immediate consequence of (2.3) is that

$$\int_R |f\tilde{g}| \, d\mu \leq \int_0^\infty f^*(t) \, g^*(t) \, dt$$

holds for every $\tilde{g} \in \mathfrak{M}_0(R, \mu)$ equimeasurable with g. This motivates the following definition.

Definition 2.20. We say that a $\sigma-$finite measure space (R, μ) is *resonant* if for every $f, g \in \mathfrak{M}_0(R, \mu)$ the following identity

$$\int_0^\infty f^*(t) g^*(t) \, dt = \sup_{\tilde{g} \in M(g)} \int_R |f\tilde{g}| \, d\mu$$

holds, where $M(g) = \{\tilde{g} \in \mathfrak{M}_0(R, \mu); \tilde{g} \text{ is equimeasurable with } g\}$.

Thankfully, resonant measure spaces have a simple characterisation.

Theorem 2.21. *A $\sigma-$finite measure space (R, μ) is resonant if and only if (R, μ) is either non-atomic or completely atomic, with all atoms having the same measure.*

Remark 2.22. The condition for a completely atomic $\sigma-$finite measure space to have all atoms with the same measure in order to be resonant is indeed necessary. Let (R, μ) consist of precisely two atoms a and b such that $\mu(\{a\}) = 1$ and $\mu(\{b\}) = 2$ and set $f = \chi_{\{a\}}$ and $g = \chi_{\{b\}}$. We easily compute that $f^* = \chi_{[0,1)}$ and $g^* = \chi_{[0,2)}$. Hence $\int_0^\infty f^*(x) \, g^*(x) \, dx = 1$. Now, assume $\tilde{g} \in \mathfrak{M}_0(R, \mu)$ is equimeasurable with g. Then $\tilde{g}(a) = 0$. Indeed, if $c = |\tilde{g}(a)| > 0$, then $\mu_{\tilde{g}}\left(\frac{c}{2}\right) = 1 + d$ where $d \in \{0, 2\}$, whereas $\mu_g\left(\frac{c}{2}\right) \in \{0, 2\}$, which would contradict the assumption that g and $\tilde{g}$ are equimeasurable. Thus $\int_R |f\tilde{g}| \, d\mu = 0$. Therefore, (R, μ) is not resonant.

Now, we shall define the spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$.

Definition 2.23. The space $L^1 + L^\infty = (L^1 + L^\infty)(R, \mu)$ is the set

$$L^1 + L^\infty = \{f \in \mathfrak{M}_0(R, \mu); f = g + h, \text{ where } g \in L^1(R, \mu) \text{ and } h \in L^\infty(R, \mu)\}$$

equipped with a norm

$$\|f\|_{L^1 + L^\infty} = \inf\{\|g\|_{L^1} + \|h\|_{L^\infty}\}, \quad f \in L^1 + L^\infty,$$

where the infimum is taken over all representations $f = g+h$ of the kind described above.

The space $L^1 \cap L^\infty = (L^1 \cap L^\infty)(R,\mu)$ is exactly the set-theoretic intersection of $L^1(R,\mu)$ and $L^\infty(R,\mu)$ equipped with a norm

$$\|f\|_{L^1\cap L^\infty} = \max\{\|f\|_{L^1(R,\mu)}, \|f\|_{L^\infty(R,\mu)}\}, \quad f \in L^1 \cap L^\infty.$$

Remark 2.24. The spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$ are rearrangement-invariant Banach function spaces provided that (R,μ) is resonant.

Finally, we can precisely state our claim from the very beginning of the section.

Theorem 2.25. *Assume X is an arbitrary rearrangement-invariant Banach function space over a resonant measure space (R,μ). Then*

$$\left(L^1 \cap L^\infty\right)(R,\mu) \hookrightarrow X \hookrightarrow \left(L^1 + L^\infty\right)(R,\mu).$$

Assume that (R,μ) is a completely atomic measure space, consisting of countably many atoms each with the same measure. We observe that $\ell^1 \subseteq \ell^\infty$, which follows immediately from the well-known necessary condition for the convergence of a series. Consequently, we obtain that $\ell^1 \cap \ell^\infty = \ell^1$ and $\ell^1 + \ell^\infty \hookrightarrow \ell^\infty$. We precisely state this observation in the following corollary.

Corollary 2.26. *Assume X is a rearrangement-invariant Banach function space over a completely atomic measure space (R,μ), consisting of countably many atoms each with the same positive measure. Then*

$$\ell^1(R,\mu) \hookrightarrow X \hookrightarrow \ell^\infty(R,\mu).$$

2.5 Morrey and Campanato spaces

The theory of Morrey and Campanato spaces naturally arises from the theory of partial differential equations. Notably, they are useful for the regularity theory of PDE. The Campanato space can be viewed as an extension of the space of functions of bounded mean oscillation (the BMO space). Roughly speaking, the BMO space is a class of functions whose deviation from their means (over cubes) is bounded (see [9, Chapter 7]).

Notation 2.27. *Throughout this section, N is a natural number and μ denotes the standard $N-$dimensional Lebesgue measure on $\mathbb{R}^N$.*

Before we define Morrey and Campanato spaces, we need the following auxiliary definition.

Definition 2.28. Assume $\Omega \subseteq \mathbb{R}^N$ is a bounded domain. We define

$$\Omega_\delta = \Omega \times (0,\delta),$$

where $\delta = \operatorname{diam}\Omega$.

For $x \in \mathbb{R}^N$ and $r > 0$, we denote

$$\Omega(x,r) = \{y \in \Omega; |y-x| < r\}.$$

Now, we can define Morrey and Campanato spaces.

Definition 2.29. Let Ω and δ be as in Definition 2.28. For $\lambda \in [0,\infty)$ and $p \in [1,\infty)$, we define

$$L_M^{p,\lambda}(\Omega) = \{f \in L^p(\Omega);\ \sup_{(x,r)\in\Omega_\delta} \frac{1}{r^\lambda}\int_{\Omega(x,r)} |f(y)|^p\, dy < \infty\}.$$

For $f \in L_M^{p,\lambda}(\Omega)$, we define a norm

$$^M\|f\|_{p,\lambda} = \left(\sup_{(x,r)\in\Omega_\delta} \frac{1}{r^\lambda}\int_{\Omega(x,r)} |f(y)|^p\, dy\right)^{\frac{1}{p}}.$$

The set $L_M^{p,\lambda}(\Omega)$ equipped with the above norm is called a *Morrey space.*

Definition 2.30. Let Ω and δ be as in Definition 2.28. For $\lambda \in [0,\infty)$ and $p \in [1,\infty)$, we define

$$L_C^{p,\lambda}(\Omega) = \{f \in L^p(\Omega);\ \sup_{(x,r)\in\Omega_\delta} \frac{1}{r^\lambda}\int_{\Omega(x,r)} |f(y) - f_{x,r}|^p\, dy < \infty\},$$

where

$$f_{x,r} = \frac{1}{\mu(\Omega(x,r))}\int_{\Omega(x,r)} f(y)\, dy.$$

For $f \in L_C^{p,\lambda}(\Omega)$, we define a norm

$$^C\|f\|_{p,\lambda} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda},$$

where

$$[f]_{p,\lambda} = \left(\sup_{(x,r)\in\Omega_\delta} \frac{1}{r^\lambda}\int_{\Omega(x,r)} |f(y) - f_{x,r}|^p\, dy\right)^{\frac{1}{p}}.$$

The set $L_C^{p,\lambda}(\Omega)$ equipped with the above norm is called a *Campanato space.*

We shall restrict ourselves to domains which are “nice enough”. What “nice enough” precisely means will be made clear by the following definition. Roughly speaking, we will exclude domains which have “infinitely sharp cusps”.

Definition 2.31. A bounded domain $\Omega \subseteq \mathbb{R}^N$ is said to be *of type* $\mathcal{A}$ if there exists a positive constant A such that for all $x \in \overline{\Omega}$ and for all $r \in (0,\delta)$

$$\mu(\Omega(x,r)) \geq Ar^N,$$

where δ is as in Definition 2.28.

Example 2.32. For example, an open ball is a set of type $\mathcal{A}$. On the other hand, the domain $\{(x,y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < x^2\}$ is not of type $\mathcal{A}$.

We shall now list some properties of Morrey spaces, which will be needed in Chapter 3.

Theorem 2.33. *Let $p \in [1, \infty)$ and $\lambda \in [0, \infty)$. Assume $\Omega \subseteq \mathbb{R}^N$ is a set of type $\mathcal{A}$. Then the following three statements hold:*

1. *$L_M^{p,\lambda}(\Omega)$ is a Banach function space.*
2. *$L_M^{p,N}(\Omega) \rightleftarrows L^\infty(\Omega)$.*
3. *If $\lambda > N$, then $L_M^{p,N}(\Omega) = \{0\}$.*

As we have just done for Morrey spaces, we will also formulate some properties of Campanato spaces. Particularly, the way Campanato spaces are embedded into other function spaces is of our interest.

Theorem 2.34. *Let $p \in [1, \infty)$ and $\lambda \in [0, \infty)$. Assume $\Omega \subseteq \mathbb{R}^N$ is a set of type $\mathcal{A}$. Then the following five statements hold:*

1. *$f \in L_C^{p,\lambda}(\Omega)$ if and only if $f \in L^p(\Omega)$ and*

$$\sup_{(x,r)\in\Omega_\delta} \frac{1}{r^\lambda} \left(\inf_{c\in\mathbb{R}} \int_{\Omega(x,r)} |f(y) - c|^p \, dy \right) < \infty.$$

2. *If $0 \leq \nu \leq \lambda$, then $L_C^{p,\lambda}(\Omega) \hookrightarrow L_C^{p,\nu}(\Omega)$.*
3. *If $\lambda \in [0, N)$, then $L_C^{p,\lambda}(\Omega) \rightleftarrows L_M^{p,\lambda}(\Omega)$.*
4. *$L^\infty(\Omega) \hookrightarrow L_C^{p,N}(\Omega)$*
5. *If $\lambda \in (N, \infty)$, then $L_C^{p,\lambda}(\Omega) \rightleftarrows C^\alpha\left(\overline{\Omega}\right)$ where $\alpha = \frac{\lambda - N}{p}$.*

2.6 Weak$-L^\infty$ space

In this section, we shall define the space weak$-L^\infty$, first introduced in [3], which has proven to be useful in analysis. Particularly, it plays an important role in the modern interpolation theory (see e.g. [5]). It is often desirable to work with the space weak$-L^\infty$ instead of the space L^∞ because weak$-L^\infty$ contains L^∞ and possesses some interpolation properties which L^∞ lacks (see e.g. [4, p. 384-385]).

Definition 2.35. $W = W(R, \mu)$ denotes the set of all functions $f \in \mathfrak{M}_0(R, \mu)$ such that $f^*(t)$ is finite for each $t > 0$ and the difference $f^{**}(t) - f^*(t)$ is bounded on $(0, \infty)$. We define

$$\|f\|_W = \sup_{t>0} (f^{**}(t) - f^*(t)), \quad f \in W.$$

The set W equipped with the above mapping is called *weak-L^∞*.

Remark 2.36. Despite the notation, $\|\cdot\|_W$ is not a norm on W. In fact, W even fails to be a linear space because there exist (nonnegative) functions which are in W but whose sum is not (details can be found in [3]). It is, however, closed with respect to scalar multiplication.

One of the most fundamental results concerning the space weak$-L^\infty$ is that it is the rearrangement-invariant hull of the BMO space (recall the beginning of the previous section about Morrey and Campanato spaces), which means that weak$-L^\infty$ consists of all functions from the BMO space and all functions equimeasurable with them (see [4, Chapter 5, Theorem 7.10]).

The quantity $f^{**} - f^*$ of its own is of great interest (see [13]). It can be seen as some kind of measure of the oscillation of f^* because one can write

$$f^{**}(t) - f^*(t) = \frac{1}{t} \int_0^t f^*(s) - f^*(t)\, ds$$

for every $t \in (0, \infty)$, which has some interesting connections with the BMO space. The quantity has also a close connection to Besov-type spaces which measure smoothness of functions by means of moduli of continuity which can be viewed as "measures of the uniform continuity" of functions (an introduction to Besov spaces can be found, for example, in [4]). It also plays an important role in measuring smoothness of functions by means of rearrangements. A detailed survey on this topic is [11]. Furthermore, the quantity $f^{**} - f^*$ appears in the definition of the fairly general class of weighted function spaces denoted by $S^p(v)$ ($p \in (0, \infty)$). The space $S^p(v)$ consists exactly of all measurable functions f on $(0, \infty)$ for which

$$\left(\int_0^\infty \left(f^{**}(t) - f^*(t)\right)^p v(t)\, dt \right)^{\frac{1}{p}} \tag{2.4}$$

is finite, where v is a weight on $(0, \infty)$ (i.e. a measurable nonnegative function). An important result, which is definitely worth noting, concerning $S^p(v)$ spaces is that the well-known inequality

$$t^{-\frac{1}{n}} \left(f^{**}(t) - f^*(t)\right) \leq C \left(\nabla f\right)^{**}(t),$$

which was first obtained (in a slightly different form) in [10, Lemma 5.1] (see also [11, Lemma 3.1]), holding for every smooth function f and every $t > 0$, can be used to derive, for $p > 1$, the following inequality

$$\|t^{-\frac{1}{n}} \left(f^{**}(t) - f^*(t)\right)\|_{L^p} \leq C \|\nabla f(t)\|_{L^p}.$$

This inequality can be viewed as an embedding of a certain Sobolev space into the space $S^p\left(t^{-\frac{p}{n}}\right)$ (see [6]). As this kind of inequalities is particularly important for the theory of partial differential equations, $S^p(v)$ spaces are of interest.

Remark 2.36 indicates, however, that one must be careful when dealing with the quantity $f^{**} - f^*$. Not only does (2.4) vanish on constant functions, but, moreover, the operation $f \mapsto (f^{**} - f^*)$ is not subadditive (recall the discussion below Definition 2.17). These issues make the study of the corresponding function spaces difficult.

Chapter 3

Algebras

In this chapter, we shall address the matter of algebras for the function spaces which have been introduced in the previous chapter. We begin with a general necessary condition and a general sufficient condition. Then the combination of these two conditions will enable us to characterise when a Banach function space is equivalent to an algebra.

We shall begin with a necessary condition. We note that, for $p = 1$, a similar assertion was established in [8, Theorem 4.5].

Theorem 3.1. *Assume $X \subseteq \mathfrak{M}(R, \mu)$ is a (quasi-)normed linear space which is equivalent to an algebra. Furthermore, assume that for every $\mu-$measurable $E \subseteq R$ such that $\mu(E) < \infty$ there exists a positive constant K such that for all $f \in X$*

$$\sup_{\lambda \in (0,\infty)} \lambda \mu \left(\{x \in E; |f(x)| > \lambda\}\right)^{\frac{1}{p}} \leq K \|f\|_X, \tag{3.1}$$

for some $p \in (0, \infty)$. Then

$$X \hookrightarrow L^\infty(R, \mu).$$

Proof. Suppose X is equivalent to an algebra. That means that there exists a positive constant C such that

$$\|fg\|_X \leq C\|f\|_X\|g\|_X, \quad \forall f, g \in X. \tag{3.2}$$

Aiming for a contradiction, suppose that $X \not\hookrightarrow L^\infty(R, \mu)$. Hence there exists a function $f \in X$ such that

$$\|f\|_{L^\infty} > 2C\|f\|_X.$$

In particular, the set

$$E = \{x \in R; |f(x)| > 2C\|f\|_X\}$$

has positive measure. Moreover, we may without lost of generality assume that $\mu(E) < \infty$, as (R, μ) is $\sigma-$finite. Fix arbitrary $k \in \mathbb{N}$. Using (3.2), we get that

$$\|f^k\|_X \leq C^{k-1}\|f\|_X^k. \tag{3.3}$$

Clearly,

$$E = \{x \in E; \left|f^k(x)\right| > 2^k C^k \|f\|_X^k\}. \tag{3.4}$$

Combining (3.1) ($\lambda = 2^k C^k \|f\|_X^k$) with (3.4) and (3.3), we get that

$$2^k C^k \|f\|_X^k \mu^{\frac{1}{p}}(E) \leq K C^{k-1} \|f\|_X^k,$$

which means that

$$2^k \mu^{\frac{1}{p}}(E) \leq K C^{-1}.$$

However, this is a contradiction, for K and C are fixed constants independent of k, $\mu^{\frac{1}{p}}(E) > 0$ and $k \in \mathbb{N}$ was chosen arbitrary.

❑

Now, we will establish a sufficient condition.

Theorem 3.2. *Assume $X \subseteq \mathfrak{M}(R,\mu)$ is a (quasi-)normed linear space which has the lattice property. Furthermore, assume that*

$$X \hookrightarrow L^\infty(R,\mu). \tag{3.5}$$

Then X is equivalent to an algebra.

Proof.

Assume that (3.5) is true and so there exists a positive constant C such that

$$\|f\|_{L^\infty} \leq C\|f\|_X, \quad \forall f \in X. \tag{3.6}$$

Let $f, g \in X$. Clearly, the following pointwise estimate

$$|fg| = |f|\,|g| \leq \|f\|_{L^\infty}\,|g| \tag{3.7}$$

holds $\mu-$a.e. Therefore, using (3.6) and (3.7), we obtain

$$\|fg\|_X \leq \|f\|_{L^\infty}\|g\|_X \leq C\|f\|_X\|g\|_X,$$

since X has the lattice property. Hence X is indeed equivalent to an algebra.

❑

Remark 3.3. One can readily verify that it does not matter in the proofs of Theorem 3.1 and Theorem 3.2 whether X is a normed linear space or a quasi-normed linear space.

Before we characterise when a Banach function space is equivalent to an algebra, we need the following technical lemma, which will prove useful later on. The inequality (3.8) is, in fact, just a variation of Chebyshev's inequality.

Lemma 3.4. *Let $p \in (0,\infty)$ and assume $f \in L^p(R,\mu)$. Then*

$$\sup_{\lambda\in(0,\infty)} \lambda \mu_f^{\frac{1}{p}}(\lambda) \leq \|f\|_{L^p(R,\mu)}. \tag{3.8}$$

Proof. Set

$$F(\lambda, x) = \begin{cases} 1 & \text{if } |f(x)| > \lambda \\ 0 & \text{if } |f(x)| \leq \lambda, \end{cases}$$

for $\lambda \in (0, \infty)$ and $x \in R$. Clearly, $F(\lambda, \cdot)$ is the characteristic function of the set $\{x \in R; |f(x)| > \lambda\}$ for each $\lambda \in (0, \infty)$. We observe that μ_f (recall Definition 2.12) is obviously non-increasing and nonnegative. Hence using the Fubini theorem, we compute that

$$\begin{aligned}
\lambda^p \mu_f(\lambda) &= \mu_f(\lambda) p \int_0^\lambda s^{p-1}\, ds \leq p \int_0^\lambda \mu_f(s) s^{p-1}\, ds \\
&\leq p \int_0^\infty \mu_f(s) s^{p-1}\, ds = p \int_0^\infty s^{p-1} \left(\int_R F(s, x)\, d\mu(x) \right) ds \\
&= \int_R \left(\int_0^\infty p s^{p-1} F(s, x)\, ds \right) d\mu(x) = \int_R \left(\int_0^{|f(x)|} p s^{p-1}\, ds \right) d\mu(x) \\
&= \int_R |f(x)|^p\, d\mu(x) = \|f\|^p_{L^p(R,\mu)},
\end{aligned}$$

for each $\lambda \in (0, \infty)$. Therefore,

$$\sup_{\lambda \in (0,\infty)} \lambda \mu_f^{\frac{1}{p}}(\lambda) \leq \|f\|_{L^p(R,\mu)},$$

as we were to prove.

❑

Now, we are prepared to state and prove a characterisation when a Banach function space is equivalent to an algebra.

Corollary 3.5. *A Banach function space X over (R, μ) is equivalent to an algebra if and only if*

$$X \hookrightarrow L^\infty(R, \mu).$$

Proof. The sufficiency part is obvious, for X has the lattice property from the very definition of a Banach function space. For the necessity part, we just need to verify that (3.1) holds.

Let $E \subseteq R$ be a $\mu-$measurable set such that $\mu(E) < \infty$ and let $f \in X$. There exists a positive constant K independent of f such that

$$\int_E |f(x)|\, d\mu(x) \leq K \|f\|_X, \tag{3.9}$$

since X is a Banach function space and $\mu(E) < \infty$. Combining Lemma 3.4 $(p = 1)$ with (3.9) yields that

$$\sup_{\lambda \in (0,\infty)} \lambda \mu_f(\lambda) \leq \int_E |f(x)|\, d\mu(x) \leq K \|f\|_X,$$

where K does not depend on f, which establishes (3.1) as f and E were chosen arbitrary.

❑

It is worth mentioning explicitly the case when X is a rearrangement-invariant Banach function space over a completely atomic measure space.

Corollary 3.6. *Assume X is a rearrangement-invariant Banach function space over a completely atomic measure space (R, μ), consisting of countably many atoms each with the same positive measure. Then X is equivalent to an algebra.*

Proof. The corollary follows immediately from Corollary 2.26 and Corollary 3.5. ❑

The following simple examples demonstrate possible usage of the results which have been just derived.

Examples 3.7.

1. Assume $G \subseteq \mathbb{R}$ is a Lebesgue measurable set with non-empty interior (with respect to the standard Euclidean topology on $\mathbb{R}$). Then $L^p(G, \lambda)$ is not equivalent to an algebra for any $p \in [1, \infty)$, where λ is the standard Lebesgue measure on G.

   Indeed, there exist $a, b \in \mathbb{R}, a < b$ such that $(a, b) \subseteq G$. Set
   $$f(x) = (x-a)^{-\frac{1}{2p}} \chi_{(a,b)}(x), \quad x \in G.$$
   One can easily verify that $\|f\|_{L^p(G,\lambda)} = 2^{\frac{1}{p}}(b-a)^{\frac{1}{2p}}$. Now, we could calculate that
   $$\int_G (f^2)^p \, d\lambda = \infty$$
   and hence $f^2 \notin L^p(G, \lambda)$ but since clearly $f \notin L^\infty(G, \lambda)$, $L^p(G, \lambda)$ cannot be equivalent to an algebra by virtue of Corollary 3.5.

2. In fact, if (R, μ) is a non-atomic $\sigma-$finite measure space ($\mu(R) > 0$), then $L^p(R, \mu)$ ($p \in (0, \infty)$) is not equivalent to an algebra. Indeed, assume $m = \mu(R) < \infty$ and find $E_1 \subseteq R$ such that $\mu(E_1) = \frac{m}{2}$, as (R, μ) is non-atomic. Since $\mu(R \setminus E_1) = \frac{m}{2}$, we can find $E_2 \subseteq (R \setminus E_1)$ such that $\mu(E_2) = \frac{m}{4}$. Now, we find $E_3 \subseteq (R \setminus (E_1 \cup E_2))$ such that $\mu(E_3) = \frac{m}{8}$ and we can proceed in the obvious way by induction. Set
   $$f = \left(\sum_{n=1}^{\infty} n \chi_{E_n} \right)^{\frac{1}{p}}.$$
   Clearly,
   $$\int_R |f|^p \, d\mu = \sum_{n=1}^{\infty} \int_R n \chi_{E_n} \, d\mu = m \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty$$
   by virtue of the Lebesgue monotone convergence theorem, thus $f \in L^p(R, \mu)$. Hence $L^p(R, \mu)$ is not equivalent to an algebra, as $f \notin L^\infty(R, \mu)$. If

$p \in [1, \infty)$, then this fact comes from Corollary 3.5, as $L^p(R, \mu)$ is a Banach function space (see Example 2.11). If $p \in (0, 1)$, we cannot use Corollary 3.5. Nevertheless, the desired result follows from Theorem 3.1 (recall Lemma 3.4). Obviously, the example can be easily modified for the case when $\mu(R) = \infty$. Moreover, (R, μ) need not be strictly non-atomic. It is enough to assume that (R, μ) has a nontrivial non-atomic part.

3. For every $p \in [1, \infty)$ the sequence space ℓ^p is equivalent to an algebra (see Corollary 3.6).

Using the facts which we already know, the situation is quite straightforward for the spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$.

Proposition 3.8. *Assume (R, μ) is resonant. The space $(L^1 \cap L^\infty)(R, \mu)$ is equivalent to an algebra. The space $(L^1 + L^\infty)(R, \mu)$ is equivalent to an algebra if and only if (R, μ) is completely atomic, with all atoms having the same measure.*

Proof. Clearly,

$$\|f\|_{L^\infty(R,\mu)} \leq \|f\|_{(L^1 \cap L^\infty)(R,\mu)}, \quad f \in \left(L^1 \cap L^\infty\right)(R, \mu),$$

from the very definition of the norm on $(L^1 \cap L^\infty)(R, \mu)$. Hence $(L^1 \cap L^\infty)(R, \mu)$ is equivalent to an algebra (cf. Corollary 3.5), as $(L^1 \cap L^\infty)(R, \mu)$ is a Banach function space (recall Remark 2.24).

If (R, μ) is completely atomic, with all atoms having the same measure, $(L^1 + L^\infty)(R, \mu)$ is equivalent to an algebra by virtue of Corollary 3.6. If (R, μ) is not completely atomic, with all atoms having the same measure, then (R, μ) is non-atomic (cf. Theorem 2.21), since (R, μ) is resonant. Consider the function f from Example 3.7 (2) with $p = 1$. Then $f \in (L^1 + L^\infty)(R, \mu)$, as

$$\|f\|_{(L^1+L^\infty)(R,\mu)} \leq \|f\|_{L^1(R,\mu)},$$

but $f \notin L^\infty(R, \mu)$. Hence $(L^1 + L^\infty)(R, \mu)$ is not equivalent to an algebra (cf. Corollary 3.5), which completes the proof. ❑

Remark 3.9. The preceding proposition shows that every rearrangement-invariant Banach function space X over a resonant measure space (R, μ) contains a subspace (namely the space $L^1 \cap L^\infty$) which is equivalent to an algebra (recall Theorem 2.25). On the other hand, X is contained in a (larger in general) space (the space $L^1 + L^\infty$) and this larger space is equivalent to an algebra if and only if (R, μ) is completely atomic, with all atoms having the same measure.

We shall now focus on Morrey and Campanato spaces. We start with a simple but useful lemma about equivalent function spaces.

Lemma 3.10. *Let $X, Y \subseteq \mathfrak{M}(R, \mu)$ be (quasi-)normed linear spaces. Assume that $X \rightleftarrows Y$. Then X is equivalent to an algebra if and only if Y is equivalent to an algebra.*

Proof. There exists a positive constant C such that

$$\|fg\|_X \leq C\|f\|_X\|g\|_X, \quad f,g \in X, \tag{3.10}$$

assuming X is equivalent to an algebra. Moreover, there exist positive constants C_1 and C_2 such that

$$C_1\|f\|_X \leq \|f\|_Y \leq C_2\|f\|_X, \quad f \in X, \tag{3.11}$$

as $X \rightleftarrows Y$. Using (3.11) and (3.10), we immediately obtain that

$$\|fg\|_Y \leq C_2\|fg\|_X \leq C_2C\|f\|_X\|g\|_X \leq C_2CC_1^{-2}\|f\|_Y\|g\|_Y, \quad f,g \in X.$$

Hence Y is equivalent to an algebra.

If we exchange the role of X and Y, we can clearly follow the same lines, which completes the proof.

❑

The following lemma, which will be needed shortly, shows that L^∞ is a proper subspace of Campanato spaces for certain values of λ.

Lemma 3.11. *Let* $p \in [1,\infty)$, $\lambda \in [0,1]$ *and* $\delta > 0$. *Then* $\log|x| \in L_C^{p,\lambda}((-\delta,\delta))$.

Proof. Set

$$f(x) = \log|x|\,, \quad x \in (-\delta,\delta) \setminus \{0\}.$$

Using Theorem 2.34 (2), we may consider only the case $\lambda = 1$. Furthermore, it is sufficient to show that there exists a positive constant C such that for every $x_0 \in \mathbb{R}$ and for every $r > 0$ there exists a constant $c(x_0,r)$ such that

$$\frac{1}{r}\int_{x_0-r}^{x_0+r} |\log|x| - c(x_0,r)|^p \, dx \leq C, \tag{3.12}$$

by virtue of Theorem 2.34 (1). Substituting $y = \frac{x}{r}$, we get

$$\frac{1}{r}\int_{x_0-r}^{x_0+r} |\log|x| - c(x_0,r)|^p \, dx =$$
$$\int_{\frac{x_0}{r}-1}^{\frac{x_0}{r}+1} |\log|y| + \log r - c(x_0,r)|^p \, dy,$$

which means that we may without lost of generality assume that $r = 1$ and denote briefly $c(x_0) = c(x_0,1)$.

Let $x_0 \in \mathbb{R}$ and suppose $|x_0| \leq 2$. Set $c(x_0) = 0$. Then

$$\int_{x_0-1}^{x_0+1} |\log|x||^p \, dx \leq \int_{-3}^{3} |\log|x||^p \, dx = \|\log|x|\|^p_{L^p((-3,3))}.$$

Now, suppose $|x_0| > 2$ and set $c(x_0) = \log|x_0|$. Since $|x_0| > 2$, we have

$$|x_0| < 2(|x_0| - 1)$$

and hence

$$\log\frac{|x_0|}{|x|}<\log\frac{|x_0|}{|x_0|-1}<\log 2,\quad x\in(x_0-1,x_0+1)\setminus\{0\}. \tag{3.13}$$

In the same manner, using the fact that $|x_0|>2$ again,

$$\log\frac{|x|}{|x_0|}<\log\frac{|x_0|+1}{|x_0|}<\log\frac{3}{2},\quad x\in(x_0-1,x_0+1)\setminus\{0\}. \tag{3.14}$$

Combining (3.13) with (3.14), we obtain that

$$\left|\log\frac{|x|}{|x_0|}\right|<\log 2,\quad x\in(x_0-1,x_0+1)\setminus\{0\}.$$

and thus

$$\int_{x_0-1}^{x_0+1}\left|\log|x|-\log|x_0|\right|^p\,dx\le\int_{x_0-1}^{x_0+1}\log^p 2\,dx=2\log^p 2.$$

Therefore, the inequality (3.12) holds and so $f\in L_C^{p,1}((-\delta,\delta))$.

❑

We shall now handle Morrey spaces and, right after it, we will deal with Campanato spaces.

Theorem 3.12. *Let $p\in[1,\infty)$ and $\delta>0$. Then $L_M^{p,\lambda}((-\delta,\delta))$ is equivalent to an algebra if and only if $\lambda\ge 1$.*

Proof. On the one hand, assume $\lambda\in[0,1)$. Using Theorem 2.34 (3) and Lemma 3.11, we get that $\log|x|\in L_M^{p,\lambda}((-\delta,\delta))$. Hence combining Theorem 2.33 (1) with Corollary 3.5, $L_M^{p,\lambda}((-\delta,\delta))$ is not equivalent to an algebra.

On the other hand, assume $\lambda\in[1,\infty)$. Then the fact that $L_M^{p,\lambda}((-\delta,\delta))$ is equivalent to an algebra follows immediately from Theorem 2.33 (2 and 3) and Lemma 3.10.

❑

Since functions from $L_C^{p,\lambda}$ spaces are also in L^p from the very definition of Campanato spaces, the condition (3.1) is satisfied (recall Lemma 3.4) for these spaces. Hence we could use Theorem 3.1 in the following proof. We shall not, nevertheless, do it. Instead, we shall provide a constructive proof which, in addition, yields some extra information (see Remark 3.14).

Theorem 3.13. *Let $p\in[1,\infty)$ and $\delta>0$. Then $L_C^{p,\lambda}((-\delta,\delta))$ is equivalent to an algebra if and only if $\lambda>1$.*

Proof. Firstly, assume $\lambda\in[0,1)$. Combining Theorem 3.12 with Theorem 2.34 (3) and using Lemma 3.10, we see that $L_C^{p,\lambda}((-\delta,\delta))$ is not equivalent to an algebra.

Secondly, assume $\lambda=1$ and set

$$f(x)=\log|x|,\quad x\in(-\delta,\delta)\setminus\{0\}.$$

Then $f \in L_C^{p,1}((-\delta,\delta))$ by virtue of Lemma 3.11. Furthermore, if we set $g = \chi_{(0,\delta)}$, then $g \in L_C^{p,1}((-\delta,\delta))$, since $L^\infty((-\delta,\delta)) \hookrightarrow L_C^{p,1}((-\delta,\delta))$ (see Theorem 2.34 (4)).

Now, we shall prove that $fg \notin L_C^{p,1}((-\delta,\delta))$. Fix $r \in (0,\delta)$. We can easily compute that

$$(fg)_{0,r} = \frac{1}{2r}\int_{-r}^{r} f(y)g(y)\,dy = \frac{1}{2r}\int_0^r \log|y|\ dy = \frac{\log r - 1}{2}.$$

Hence

$$\frac{1}{r}\int_{-r}^{r} |f(y)g(y) - (fg)_{0,r}|^p\ dy \geq \frac{1}{r}\int_{-r}^{0}\left|\frac{\log r - 1}{2}\right|^p\ dx = \left|\frac{\log r - 1}{2}\right|^p. \tag{3.15}$$

Therefore, $fg \notin L_C^{p,1}((-\delta,\delta))$, since the right side of (3.15) clearly tends to infinity as r approaches 0.

Finally, assume $\lambda > 1$. Considering Theorem 2.34 (5) and Lemma 3.10, it is sufficient to show that $C^\alpha\left(\overline{(-\delta,\delta)}\right)$ is equivalent to an algebra where $\alpha = \frac{\lambda-1}{p}$. Let $f, g \in C^\alpha\left(\overline{(-\delta,\delta)}\right)$. We straightforwardly compute that

$$\begin{aligned}
\|fg\|_\alpha = \|fg\|_\infty + \sup_{\substack{x,y\in(-\delta,\delta)\\ x\neq y}} \frac{|f(x)g(x) - f(y)g(y)|}{|x-y|^\alpha} \leq \\
\|f\|_\infty\|g\|_\infty + \sup_{\substack{x,y\in(-\delta,\delta)\\ x\neq y}} \frac{|f(x)|\,|g(x)-g(y)| + |g(y)|\,|f(x)-f(y)|}{|x-y|^\alpha} \leq \\
\|f\|_\infty\|g\|_\infty + \|f\|_\infty \sup_{\substack{x,y\in(-\delta,\delta)\\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|^\alpha} + \|g\|_\infty \sup_{\substack{x,y\in(-\delta,\delta)\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq \\
\|f\|_\alpha\|g\|_\alpha,
\end{aligned}$$

which means that $C^\alpha\left(\overline{(-\delta,\delta)}\right)$ is equivalent to an algebra. This completes the proof of the theorem.

❑

Remark 3.14. Not only have we shown in the course of the above proof that $L_C^{p,1}((-\delta,\delta))$ is not equivalent to an algebra, but we also showed that even the product of a bounded function and a function from $L_C^{p,1}((-\delta,\delta))$ needn't be in $L_C^{p,1}((-\delta,\delta))$.

Even though weak$-L^\infty$ fails to be a linear space (see Remark 2.36), thus Definition 2.6 does not apply to it, it is reasonable to address the question whether weak$-L^\infty$ is closed under pointwise multiplication or not. Moreover, even the inequality (2.1) from Definition 2.6 makes perfect sense provided that we replace the (quasi-)norm with the mapping $\|\cdot\|_W$. Hence we shall use the term algebra even for weak$-L^\infty$, albeit it is not formally correct.

Theorem 3.15. $W(0,1)$ *is not equivalent to an algebra.*

Proof. Set

$$f(x) = \log x, \quad x \in (0,1).$$

Firstly, we observe that $f \in W(0,1)$. Indeed, we have

$$\mu_f(\lambda) = \mathrm{e}^{-\lambda}, \quad \lambda \in [0,1],$$

from the very definition of $\mu_f(\lambda)$ and thus

$$f^*(t) = \log \frac{1}{t}, \quad t \in (0,1). \tag{3.16}$$

Using (3.16), we compute that

$$f^{**}(t) = \frac{1}{t} \int_0^t \log \frac{1}{y} \, dy = 1 - \log t, \quad t \in (0,1).$$

Hence $f \in W(0,1)$, for

$$\|f\|_W = \sup_{t>0} \left(f^{**}(t) - f^*(t)\right) =$$
$$\sup_{t \in (0,1)} \left(1 - \log t - \log \frac{1}{t}\right) = 1.$$

We shall now prove that $f^2 \notin W(0,1)$. Indeed, we easily compute that

$$\mu_{f^2}(\lambda) = \mathrm{e}^{-\sqrt{\lambda}}, \quad \lambda \in [0,1],$$

and

$$\left(f^2\right)^*(t) = \log^2 t, \quad t \in (0,1).$$

Whence

$$\left(f^2\right)^{**}(t) = \frac{1}{t} \int_0^t \log^2 y \, dy = \log^2 t - 2\log t + 2, \quad t \in (0,1).$$

Thus

$$\|f^2\|_W = \sup_{t \in (0,1)} \left(\log^2 t - 2\log t + 2 - \log^2 t\right) = \infty.$$

Hence $f^2 \notin W(0,1)$.

❑

Chapter 4

Conclusion

As expected, the results obtained in Chapter 3 suggest that it is rather rare for a function space to be equivalent to an algebra. Theorem 3.1 entails (under certain mild conditions) that should a function space be equivalent to an algebra, it is necessary that it is embedded into L^∞. Not surprisingly, if a function space is "rich enough", it usually also contains essentially unbounded functions. Hence Theorem 3.1 significantly reduces the class of function spaces which potentially might be equivalent to an algebra. On the other hand, when a function space is indeed embedded into L^∞, then it is necessarily equivalent to an algebra provided that it has the lattice property, as Theorem 3.2 shows.

The assertion of Theorem 3.1 includes an extra, rather technical, assumption (3.1), which is used in the proof of the theorem. That assumption is, however, not overly restrictive, as it is satisfied, e.g., when the functions from the given function space are integrable over sets of finite measure (recall Lemma 3.4). Yet it is possible that the extra condition is unnecessary and the conclusion of the theorem holds without it. However, in that case a different proof would be required.

If we thoroughly check the proofs of Theorem 3.1 and Theorem 3.2, we see that neither the fact that the spaces are closed with respect to function addition nor the triangle inequality from the definition of a norm (or its analogue from the definition of a quasinorm) is actually needed. Hence we could develop the theory in a more general fashion. Instead of X being a linear subspace of $\mathfrak{M}\,(R,\mu)$, we could assume only that X is a subset of $\mathfrak{M}\,(R,\mu)$ which is closed with respect to scalar multiplication, that is, whenever f is an element of X, then αf is also in X for every $\alpha \in \mathbb{R}$ (note that it means in particular that $0 \in X$). Then, instead of a norm or a quasinorm on X, we would consider X to be equipped only with a functional $\rho : X \to [0,\infty)$ with the following two properties:

1. For every $f \in X : \rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.

2. For every $f \in X$ and each $\alpha \in \mathbb{R} : \rho(\alpha f) = |\alpha|\, \rho(f)$.

Such a functional could be, for an obvious reason, called a *norm-potential functional.* Under these assumptions, the theorems would hold even for more general class of function spaces provided that we would also modify the corresponding definitions appropriately. For instance, Theorem 3.1 would also be applicable to weak$-L^\infty$.

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